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# Steady-state temperature in a roll subject to surface heating and convection cooling

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**Abstract.** An iterative procedure, in which an analytical solution known for a simpler problem is used to obtain the solution to a problem with much more complicated non-linear boundary conditions, is presented. Sufficient convergence conditions in S.L. Sobolev's space $W_2^1(A)$ are established. This algorithm can be useful when fast solutions are required, for example, in controlling an optimized cooling system for hot metal rolling. An example is given.

## 1. Introduction

The steady-state temperature analysis of a rotating cylinder has attracted the attention of many researchers in view of important applications of this model to metal-sheet rolling. The heat flux into the cylinder is due to the direct contact of the roll with the metal sheet, which has a higher temperature. Since the temperature gradients in the contact area are predominantly radial, the heat fluxes are calculated with the assumption of one-dimensional flow resulting from a sudden contact of two bodies with different temperatures. Additional heat flux is generated at the contact area due to friction and plastic deformation of the rolled metal [1–5]. A detailed model for the heat-flux calculations in the contact area is given in [5]. A number of attempts have been made to analyze the influence of the cooling systems on the temperature distributions on the surface of the roll. In analytical approaches, the boundary conditions were simplified to make the analysis tractable. For example, different analytical techniques were used [3, 4, 6, 7] to obtain the solution to the problem with the constant heat flux in the contact area, and the convective cooling,

$$K\, \partial T/\partial r = h(T_c - T)\,, \tag{1}$$

outside the contact with $T_c$, $K$, and $h$ being a coolant temperature, conductivity, and a constant. In fact, however, $h$ varies in circumferential direction, having high values in the area of spraying, and low values in the area of air cooling. Moreover, there is considerable evidence [8, 9] that at each given point in the spraying area, the coefficient $h$ is not a constant, but depends on the temperature of the roll. While the scheme with a constant $h$ probably gives some average representation of the temperature for a given cooling system, it does not change when sprayers are moved along the surface. Attempts to model more realistic boundary conditions numerically are known in the literature. Because of the high-temperature gradients, the grid/mesh should be refined. This creates an enormous volume of calculations. Bennon [10], for example, attacked a three-dimensional problem, considering variation of the surface temperature in both axial and circumferential directions. Computing was extensive; 13,000 nodes were involved. Tseng [11] used a first-order upwind scheme to study the heat-transfer behavior of a two-dimensional rotating roll. The upwind technique leads to a stable numerical process, but it often produces an artificial diffusion comparable with the real conduction; and, therefore, has poor accuracy. The author remarks correctly that the case of fast rotation is favorable to this technique and a few results for $h$ varying in the circumferential direction (but not dependent on the temperature) were obtained. This approach generates a large system of equations, the solution to which can be obtained directly or by iteration. In the case $h = f(T_c - T)$, this creates difficulties.

In this paper, we present an iterative algorithm which uses solutions for a problem with simple boundary conditions – for which the analytical solution is trivial – to obtain the solution to the problem with the required boundary conditions. This includes the case of $h$ dependent on both surface position and temperature. We establish the convergence of the algorithm, and evaluate the rate of convergence directly in terms of the process parameters. We also demonstrate the efficiency of the iteration on the examples.

## 2. Governing equations

If the coordinate system is fixed with space and does not rotate with the roll, governing equation for the steady state temperature distribution can be written

$$\Delta T - (\omega/k)\frac{\partial T}{\partial \theta} = 0\,, \tag{2}$$

where $T = T(r, \theta)$, $\Delta$ is the Laplace operator in the polar coordinates $r$ and $\theta$, $k$ is the diffusivity, and $\omega$ is the angular velocity.

Boundary conditions in the area of contact with hot metal can be modeled by assigning the heat flux

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = f(\theta)\,, \tag{3}$$

where $r$ and $R$ are current and outside radii of the roll. At the remainder of the surface, conditions are modeled by Eq. (1).

If the heat flux at the surface of the roll were given, it could have been presented in a form of the Fourier series

$$T_n = \sum_{m=-\infty}^{\infty} A_m \exp(im\theta)\,, \tag{4}$$

where the term $m = 0$ is omitted and $T_n$ is a shorthand notation for $\partial T/\partial r|_{r=R}$. The temperature in the roll could have been expressed as

$$T(r, \theta) = A_0 + \sum_{m=-\infty}^{\infty} A_m J_m(\gamma_m r)/\{\gamma_m J_m'(\gamma_m r)|_{r=R}\} \exp(im\theta)\,, \tag{5}$$

where $J_m$ and $J_m'$ denote the Bessel function of order $m$ and its derivatives with respect to its argument $\gamma_m r$, $\gamma_m = \sqrt{-im\omega/k}$, and $i = \sqrt{-1}$. Expression (5) is a general form of the solution to Eq. (2). This can be directly verified by substitution.

Remarks for future use are:

1. The term $A_0$ is the average temperature on the surface. It is not defined by heat flux (4), but is found from the condition that temperature (5), substituted for $T$ in the right hand side of Eq. (1) and combined with expression (3), generates a balanced heat flux across the surface,

$$\oint T_n \, ds = 0 . \tag{6}$$

2. Because the temperature and the heat flux are real, pairs of terms, corresponding to $m$ and $-m$ in Eqs (4) and (5) have to be reciprocally conjugate.
3. The motivation for this work was its application to hot metal rolling where practically possible values of $R\sqrt{(\omega/k)}$ are on the order of $10^3$. This justifies the use of asymptotic approximation for the Bessel's functions in Eq. (5), giving

$$T(r, \theta) = A_0 + \sum_{m=-\infty}^{\infty} A_m T_m(r) \exp(im\theta) , \tag{7}$$

where

$$T_m(r) = (1 - i) \exp[p_m(1 + i)(r - R)]/(2p_m) , \tag{8}$$

and

$$p_m = \sqrt{m\omega/(2k)} . \tag{9}$$

The presence of exponential terms with respect to $r - R$ with a large multiplier, $p_m$, shows that indeed the surface temperature variations quickly decay with depth.

Below, we develop an iterative procedure in which the expressions (4) and (5), or (4) and (7) are used to obtain a practically analytical solution to the problem. We establish convergence conditions, and give a numerical example.

## 3. Description of the algorithm

Letting superscripts denote the iteration number, we start with a current iteration for a heat flux across the surface of the roll, $T_n^{(j)}$. This heat flux should comply with the given distribution in the area of contact and should meet condition (6), understood for a steady state. Aside from the aforementioned conditions, the form of the heat flux is arbitrary.

The iterative process can be formulated as follows.

1. Present the heat flux in form (4) and use Eq. (5) to calculate the temperature, with $A_0$ as yet unknown.
2. Use boundary conditions, Eq. (3), and substitute $T^{(j)}$ for $T$ in the right hand side of Eq. (1) to obtain the heat flux for the next iteration, $j + 1$. Choose $A_0$ to comply with condition (6).
3. Go to step 1.

The current ($j$) iteration for temperature and heat flux distribution is treated as an element $\mathbf{T}^{(j)}$ of a function space in one of the norms defined in Section 4. Then we can write

$$\mathbf{T}^{(j+1)} = P\mathbf{T}^{(j)}\,, \tag{10}$$

where $P$ is an operator of the iterative procedure, and the bold face denotes the elements of a space. We also consider a modified iterative process

$$\mathbf{T}^{(j+1)} = Q\mathbf{T}^{(j)}\,, \qquad Q \equiv (1-\mu)I + \mu P\,, \tag{11}$$

where $I$ is the identity operator. The procedure defined by (11) shows that a subsequent iteration is obtained as a linear combination of the previous iteration and the new iteration delivered by the process (10) with weights, respectively, $1-\mu$ and $\mu$. Obviously, for $\mu = 1$ process (11) degenerates into (10).

In the following section, we show that iterative process (11) can be made convergent by an appropriate choice of $\mu$.

## 4. Convergence; linear problem

We start with the problem where $h$ depends on the circumferential coordinate alone and does not depend on temperature. Then the problem becomes linear, and convergence does not depend on the right hand side of all equations. We will assume in this section that $T_c = 0$ and $T_n = 0$ in the area of contact. We start with a non-zero approximation. If in the considered iterative process it converges to zero, then for the problem with $T_n \neq 0$ and $T_c \neq 0$ it converges to the required solution.

This statement is useful for our purposes:

LEMMA 1. *If* $(\mathbf{T}, P\mathbf{T}) < B(\mathbf{T}, \mathbf{T})$, $\forall \mathbf{T}$, $B < 1$, *and operator $P$ is bounded, then there exists such a sufficiently small* $\mu$, $0 \leq \mu < 1$, *that iteration* (11) *converges.*

*Proof.* It is sufficient to show that

$$(Q\mathbf{T}, Q\mathbf{T})/(\mathbf{T}, \mathbf{T}) \leq 1\,, \quad \forall \mathbf{T}\,. \tag{12}$$

By definition,

$$\begin{aligned}(Q\mathbf{T}, Q\mathbf{T}) &= (\mu P\mathbf{T} + (1-\mu)\mathbf{T}, \mu P\mathbf{T} + (1-\mu)\mathbf{T}) \\ &= (1 - 2\mu + \mu^2)(\mathbf{T}, \mathbf{T}) + 2\mu(1-\mu)(\mathbf{T}, P\mathbf{T}) + \mu^2(P\mathbf{T}, P\mathbf{T})\,.\end{aligned} \tag{13}$$

Expression (13) implies

$$\begin{aligned}(Q\mathbf{T}, Q\mathbf{T})/(\mathbf{T}, \mathbf{T}) &\leq (1 - 2\mu + \mu^2) + 2\mu(1-\mu)(\mathbf{T}, P\mathbf{T})/(\mathbf{T}, \mathbf{T}) + \mu^2\|P\|^2 \\ &\leq (1 - 2\mu) + 2\mu(\mathbf{T}, P\mathbf{T})/(\mathbf{T}, \mathbf{T}) + O(\mu^2) \leq 1 - 2\mu(1-B) + O(\mu^2)\,.\end{aligned} \tag{14}$$

By definition, $1 - B$ is positive, and $O(\mu^2)$ can be discarded if $\mu$ is small. This shows that (12) is true if $\mu$ is sufficiently small.

Lemma 1 is valid for arbitrary inproduct and norm. We will show now that for the considered process, the conditions of the lemma are met in the Sobolev norm $W_2^1(A)$, see [13].

We define a function space by introducing a scalar product and a norm:

$$(\mathbf{u}, \mathbf{v}) = \oint (uv_n + vu_n)\,\mathrm{d}s + 2\oint' \alpha \mathbf{uv}\,\mathrm{d}s\,, \tag{15}$$

$$\|\mathbf{u}\|^2 = (\mathbf{u}, \mathbf{u})\,. \tag{16}$$

In Eq. (15), prime shows that the contact area is omitted, $\alpha = h/K$, and as before, $u_n$ is a shorthand notation for $\partial u/\partial r|_{r=R}$. It can be shown by integration by parts that

$$(\mathbf{u}, \mathbf{u}) = 2\Big(\oint uu_n\,\mathrm{d}s + \oint' \alpha u^2\,\mathrm{d}s\Big) = 2\Big(\int \mathrm{grad}(u)^2\,\mathrm{d}A + \oint' \alpha u^2\,\mathrm{d}s\Big) > 0\,, \quad \mathbf{u} \neq 0\,. \tag{17}$$

Therefore, norm (16) is equivalent to that of the Sobolev's space $W_2^1(A)$. We also introduce an auxiliary scalar product and norm by

$$(\mathbf{u}, \mathbf{v})_1 = \oint (uv_n + vu_n)\,\mathrm{d}s\,, \tag{18}$$

$$\|\mathbf{u}\|_1^2 = (\mathbf{u}, \mathbf{u})_1\,. \tag{19}$$

It follows from the embedding theorems that this norm is also equivalent to norm (16). We can further reduce form (15) to

$$(\mathbf{u}, \mathbf{v}) = \oint (uv_n + vu_n + 2\alpha uv)\,\mathrm{d}s\,, \tag{20}$$

by artificially assuming that $\alpha = 0$ in the area of contact.

We treat a temperature and all its derivatives, including the normal derivatives on the boundary, obtained for an iteration $(j)$, as an element $\mathbf{T}^{(j)}$ of the Sobolev function space in the norm defined above. Moreover, all iterations belong to the subspace of solutions to Eq. (2), and therefore mapping $P$ is a mapping from this subspace into itself. In contrast to a more common procedure, we start with the normal derivative at $r = R$ and then uniquely reconstruct the temperature. A fine point here is that although the constant $A_0$ for the element of the set $\mathbf{T}^{(j)}$ is defined from the substitution for the current temperature $T^{(j)}$ in Eq. (1), the heat flux itself generated this way already belongs to the element $\mathbf{T}^{(j+1)}$.

LEMMA 2. *In the scalar product and the norm given by* (15, 16)

$$(\mathbf{u}, P\mathbf{u}) - 0.5(\mathbf{u}, \mathbf{u}) \leq 0\,. \tag{21}$$

*Proof.* Using the expression

$$(Pu)_n = -\alpha u\,, \tag{22}$$

following from the definition of the iterative process, we can write:

$$\begin{aligned}(\mathbf{u}, P\mathbf{u}) - 0.5(\mathbf{u}, \mathbf{u}) &= \oint \{u(Pu)_n + (Pu)u_n + 2\alpha u(Pu)\}\,\mathrm{d}s - \oint (uu_n\,\mathrm{d}s + \alpha u^2)\,\mathrm{d}s \\ &= \oint \{[u(Pu)_n + (Pu)u_n - ((Pu)_n(Pu) - \alpha u(Pu))] - [uu_n\,\mathrm{d}s - u(Pu)_n]\}\,\mathrm{d}s\end{aligned}$$

$$= \oint \{[u(Pu)_n - (Pu)_n(Pu)] + [u(Pu)_n + (Pu)u_n - (Pu)_n(Pu) - uu_n]\}\, ds$$

$$= \oint \{[-\alpha u^2 - (Pu)_n(Pu)] + [u(Pu)_n + (Pu)u_n - (Pu)_n(Pu) - uu_n]\}\, ds$$

$$= -\oint \alpha u^2\, ds - 0.5(P\mathbf{u}, P\mathbf{u})_1 - 0.5[-2(\mathbf{u}, P\mathbf{u})_1 + (P\mathbf{u}, P\mathbf{u})_1 + (\mathbf{u}, \mathbf{u})_1]$$

$$= -\oint \alpha u^2\, ds - 0.5(P\mathbf{u}, P\mathbf{u})_1 - 0.5(P\mathbf{u} - \mathbf{u}, P\mathbf{u} - \mathbf{u})_1 \leqslant 0\,. \tag{23}$$

This means that lemma 2 is correct and that the first condition in lemma 1 is met with $B = 0.5$. We repeat here for the sake of clarity that the bold face denotes the elements of the spaces, while the plain text denotes the value of a function for the current coordinates $r$ and $\theta$.

LEMMA 3. *Operator $P$ is bounded.* This will be shown in the following section in the course of investigation for the non-linear problem by direct application of expressions (4) and (5).

The result of this section can be formulated:

THEOREM. *The iterative process* (10) *is convergent in norm* (16) *with the parameter $\mu$ taken sufficiently small.*

## 5. Convergence; non-linear problem

In a non-linear problem convergence depends on an initial approximation. Suppose that we already have an approximation in the neighborhood of the solution, and the suggested algorithm is a way to improve it. We also assume that the expression $\alpha = \alpha(T)$ is reasonably well-behaved, for example, it is non-negative, continuous and bounded in the neighborhood of solution. This means among other things that $\partial T/\partial r$ on the surface is uniquely identified by Eq. (1) when $T$ is given, and vice versa, $T$ is unique when $\partial T/\partial r$ is given. In further derivations, $T_c$ is taken as a reference point. Then other implications of these assumptions can be written as:

$$-\,\mathrm{d}[\alpha(T)T]/\mathrm{d}T \geqslant 0\,; \qquad T(r, \theta) \geqslant 0\,. \tag{24}$$

Violation of the first condition in (24) would mean that the heat flux from the roll to coolant decreases as the temperature difference at the interface increases. Violation of the second condition would result in the temperature of the roll being lower than the coolant temperature in the neighborhood of solution. Both are physically unreasonable. These conditions cover a wide variety of descriptions, including radiative heat transfer with $\partial T/\partial n = bT^4$, since it is equivalent to (1) with $\alpha = h/K = bT^3$, $T_c = 0$.

The scalar product and the norm defined by (15) and (16) are not applicable for non-linear case, because $\alpha$ itself depends on the iteration. However, under the described conditions, we still can use norm (19) or either of the following scalar products and corresponding norms

$$(\mathbf{u}, \mathbf{v})_2 = \oint (v_n)(u_n)\, \mathrm{d}s , \tag{25}$$

$$(\mathbf{u}, \mathbf{v})_3 = \oint (uv)\, \mathrm{d}s , \tag{26}$$

to establish convergence. Suppose, for example, that the surface temperature is convergent in the Hilbert space $L_2$, as given by (26). Then, because of Eq. (1), the process is convergent in norm $\|\mathbf{u}\|_2$, and due to Schwarz's inequality ([14], p. 11) it is also convergent in $\|\mathbf{u}\|_1$. Using the same reasoning, we conclude that if the process is convergent in the norm defined by (25), it is also convergent in the norm given by (26). Further, we will use the scalar product (25) with subscript 2 being omitted.

Suppose, a current approximation, $j$, is given by expressions (4) and (5) or (4) and (8). Let the exact solution to the problem be given by Eqs (4) and (7). Let also a current iteration, $j$, be

$$T_n^{(j)} = \sum_{m=-\infty}^{\infty} (A_m + \delta A_m) \exp(im\theta) , \tag{27}$$

$$T(r, \theta)^{(j)} = \sum_{m=-\infty}^{\infty} (A_m + \delta A_m)(1 - i)/(2p_m) \exp[p_m(1 + 1)(r - R) + im\theta] . \tag{28}$$

Since $P\mathbf{T} = \mathbf{T}$, where $\boldsymbol{T}$ is the true solution, by definition, the iterative process converges if

$$\|P\mathbf{T}^{(j)} - P\mathbf{T}\| / \|\mathbf{T}^{(j)} - \mathbf{T}\| < 1 . \tag{29}$$

Orthogonality conditions

$$\int \exp(im\theta) \exp(ik\theta)\, \mathrm{d}\theta = \{0 \text{ if } m \neq -k;\ 2\pi \text{ if } m = -k\} , \tag{30}$$

yield

$$\|\mathbf{T}^{(j)} - \mathbf{T}\|^2 = \int \sum_{m=-\infty}^{\infty} \delta A_m \exp(im\theta) \sum_{k=-\infty}^{\infty} \delta A_k \exp(ik\theta) R\, \mathrm{d}\theta = 4\pi R \sum_{m=1}^{\infty} |\delta A_m|^2 . \tag{31}$$

The left hand side of inequality (29) is evaluated with the assumption that all $\delta A_m$ are small. Therefore, all the corresponding deviations from the exact solution in temperature, heat fluxes and $\alpha$, are small and can be approximated with one linear term of Taylor's expansion. Then

$$\begin{aligned} PT_n^{(j)} - T_n &= -\{\mathrm{d}[\alpha(T)T]/\mathrm{d}T\}\, \delta T \\ &= -\{\mathrm{d}[\alpha(T)T]/\mathrm{d}T\}\Big\{\delta A_0 + \sum_{m=-\infty}^{\infty} \delta A_m(1 - i)/(2p_m) \exp(im\theta)\Big\} , \end{aligned} \tag{32}$$

from Eq. (1), and from the definition of the operator $P$. Therefore,

$$\|P\mathbf{T}^{(j)} - P\mathbf{T}\| \leq 4\pi R\Big\{\frac{\delta A_0^2}{2} + \sum_{m=-1}^{\infty} |\delta A_m(1 - i)/(2p_m)|^2\Big\}^{1/2} \max\{\mathrm{d}[\alpha(T)T]/\mathrm{d}T\} . \tag{33}$$

In this transition from (32) to (33), inequalities (24) were used.

The upper bound for ratio (29) can be evaluated by retaining just one, the least favorable, term in sums (31) and (33). This happens for $m = 1$, giving the convergence criteria in terms of rolling process parameters

$$\max\{\mathrm{d}[\alpha(T)T]/\mathrm{d}T\}/p_1 = \max\{\mathrm{d}[\alpha(T)T]/\mathrm{d}T\}\sqrt{2k/\omega} < 1 . \tag{34}$$

In the case of linear problem $\alpha$ does not depend on the temperature. Therefore, iterations (10) converge if

$$\max(\alpha)\sqrt{2k/\omega} < 1 . \tag{35}$$

The following remark should be made. The increment $\delta A_0$ in Eq. (32) is not independent, but should follow the increments of $\delta A_m$ in order to maintain the balanced heat flux across the surface of the cylinder. However, accounting for this additional term in ratio (29) introduces small changes in inequality (34) of an order of $1/p_1$, and therefore, it can be neglected.

## 6. Discussion

The following remarks are intended to place the present approach in a proper perspective in relationship with other papers on this topic.

6.1. Some useful insight related to the asymptotic solution of Eqs (7) and (8) can be obtained by treating a steady-state problem for a half-plane subjected to a heat flux, with a period $2\pi R$ moving with the speed $\omega R$ along the surface, $y = 0$. The governing equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{\omega R}{k}\frac{\partial T}{\partial x} = 0 . \tag{36}$$

Let the heat flux on the surface be given as a Fourier series

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = \sum_{m=-\infty}^{\infty} A_m \exp(imx/R) . \tag{37}$$

Then the solution of (36) is

$$T(x, y) = A_0 - \sum_{m=-\infty}^{\infty} (A_m/b_m)\exp(-b_m y + imx/R) , \tag{38}$$

where

$$b_m = \frac{1}{R}\sqrt{m}\sqrt{m - i\omega R^2/k} , \tag{39}$$

and as before, the terms with $m = 0$ in (37) and (38) are omitted. The value of $\omega R^2/k$ in (39) is in the range $0.5 * 10^6$ to $5 * 10^6$ for all practical rolling processes. Even if we take 1000 terms in the Fourier series, $m$ is still a small number in the second radical in (39) and can be omitted. Then (38) becomes identical to (8). Formally, this approximation results from neglecting the first term in (36). Therefore, the asymptotic expression (8) is the solution for a cylinder unwrapped into a half-plane.

6.2. In papers [3, 4] the form of solution is similar to that given by (5), although the coefficients are different, because the boundary conditions here are more complicated. In principle, this form could be used to obtain the solution of the problem with the required boundary conditions, but it would lead to a system of non-linear algebraic equations which is difficult to solve. Even in the linear case, the matrix is neither diagonally dominant nor positive definite, and the round-off error could be significant for a sizable problem. One may view the present algorithm as an iterative procedure of solving these equations. We also note that the authors of papers [3, 4] presented the formulae in the form with real and imaginary parts being separated, yielding much longer expressions with Kelvin's functions Ber and Bei. Probably, it did not provide any advantages because the subroutines for the Bessel functions of a complex argument are more readily available than the subroutines for Ber and Bei functions.

6.3. In finite difference/element approaches like that in [9, 12], the volume of calculations is prohibitive. Since the radial gradients predominate, the problem is most often treated as a sequence of one-dimensional calculations with the boundary conditions updated as the "beam" rotates and enter different cooling areas. The steady-state solution is treated as the transient problem limit. As Tseng notes [11], at least 240 steps are needed to simulate one revolution. Hundreds of thousands of steps are needed to reach the steady-state. We add, that as a result of round-off error accumulation, and the neglected gradients in the circumferential direction, the temperature at the same point will change from revolution to revolution, and probably the steady state cannot be reached at all. To great extent, this problem has been overcome in view of fast developments in the power of contemporary computers. However, in certain cases, when fast decisions are needed, for example, in a control system, the present algorithm could be useful.

6.4. Finally, we comment on our experience with the present algorithm. Although it is predominantly analytical, the numerical harmonic analysis was used to decompose the given heat flux into the Fourier series (5). For the set of problems we solved, 200–400 points were sufficient for accurate representation. Since only summation without solving the system of equations was needed, a 10 or even 100 fold increase in the number of terms and integration points would not be prohibitive, if it were required.

Evaluations (34) and (35) or conservative. We found, that in many cases, the process (11) was still convergent with $\mu = 1$ and these conditions not being met. As expected in the linear cases, the process was always convergent regardless of parameters of rolling and initial approximations. After the appropriate choice of $\mu$ had been made, it took a few seconds to solve the problem on a medium size computer, a VAX in our case. We started with $\mu = 1$, and a satisfactory value was obtained after 2–3 attempts by dividing the current trial value by 2. In general, convergence for the non-linear case depends on the initial approximation. For this process, however, we failed to find a set of parameters and initial approximations leading to a divergent process.

Our experience shows that for the parameters in a practical range convergence with three decimal places can be obtained in 2 to 30 iterations. In the latter case the coefficient $\mu$ had to be reduced to 0.125.

## 7. Example

We consider temperature distribution in a roll, $k = 0.02\,\mathrm{m}^2/\mathrm{hr}$, $R = 0.3\,\mathrm{m}$, angular velocity $\omega = 1$ revolution per second, with the following boundary conditions.

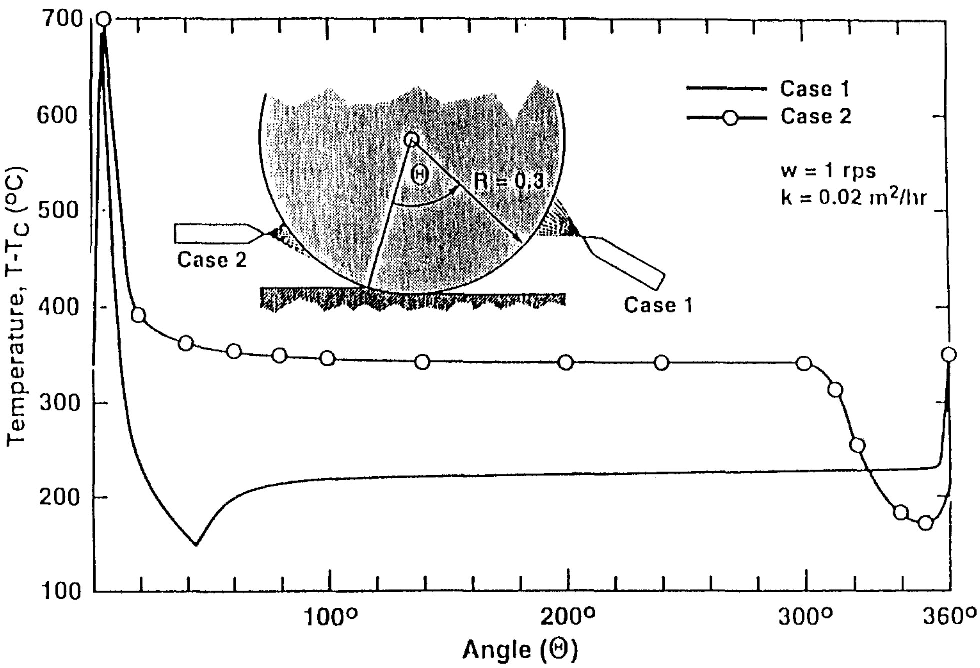


*Fig. 1.* Surface temperature distribution for two versions of cooling.

Contact area: $0 \leq \theta \leq 4.5°$; $\partial T/\partial n = q/K = 1.415 * 10^6$, °C/m.
Cooling area: $10° \leq \theta \leq 46°$; $\partial T/\partial n = \alpha(T_c - T) = [(1000 + 0.1(T - T_c)](T_c - T)$, $T$ and $T_c$ in °C, and $\alpha$ in $1/m$. [Outside the specified cooling area, $\partial T/\partial n = 0$.] The results are given in Fig. 1 by the solid line. The line with circles shows the temperature distribution when the cooling system is moved on the other side with respect to the contact area, $314° \leq \theta \leq 350°$. The average temperatures, respectively, are 229°C and 336.3°C. The difference suggests that the idea of averaging $\alpha$ over the roll surface is not very useful.

Finally, Table 1 demonstrates the convergence of surface temperatures. This table was generated by inspection of temperatures in any two subsequent iterations, finding the point with the largest difference, and recording the absolute value of this difference. We can see that for the presented example, seven iterations, obtained with $\mu = 0.5$, are sufficient for any practical purposes.

Figure 2 shows more detail in the temperature distribution in the neighborhood of the contact area. As expected, the temperature variations quickly decay with depth. We also

*Table 1.* Surface temperature convergence

| Iteration number | Maximum temperature difference for two subsequent iterations |
|---|---|
| 1 | 341.04 |
| 2 | 98.26 |
| 3 | 4.66 |
| 4 | 1.56 |
| 5 | 0.38 |
| 6 | 0.05 |
| 7 | 0.006 |

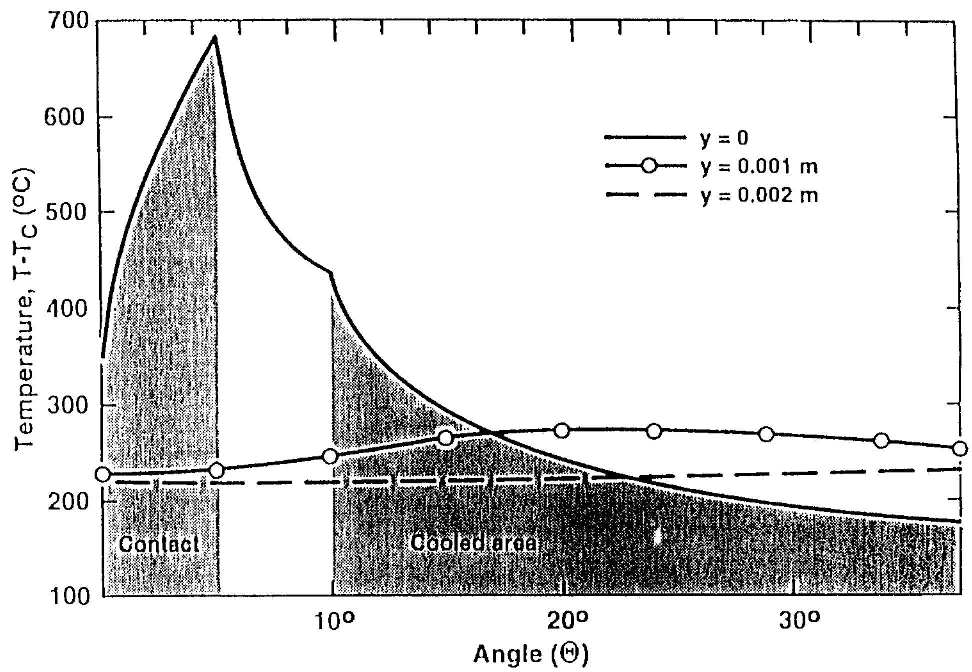


*Fig. 2.* Surface and subsurface temperature distribution.

note, that for a fixed depth, the maximum temperature is not under the contact area, and moves in the direction of rotation with depth.

## 8. Conclusions

In this paper, we presented an iterative algorithm which leads to a practically analytical solution to a problem of rotating cylinder with complicated, including non-linear, boundary conditions. Sufficient convergence conditions are given. This algorithm turns out to be numerically efficient in applications to metal roll modeling, and is a feasible alternative to a purely numerical scheme in cases when fast calculations are crucial, for example, in developing a cooling control system. The developed algorithm is intended to use for analysis of the cyclic thermal stresses in the roll, thus leading to an optimal cooling for survivability of the roll. Once temperature is obtained, the stress analysis is a much simpler task, since the surface is of the simplest possible form, a circle. Results will be reported later.

## References


1. S. Cerni, A.S. Weinstein and C.F. Zorovski, The temperature and thermal stresses in the rolling of metal strip. *Iron and Steel Engineer* 40 (Sep 1963) pp. 165–171.
2. C. Troeder, A. Spielvogel and J.W. Xu, Temperature stresses in work rolls during the hot rolling of strip. *Steel Research* 56 (1985) pp. 379–384.
3. A.A. Tseng, S. Tong, P. Sun and K. Huang, Thermal behavior of aluminum rolling. ASME Proceeding of 1988 National Heat Transfer Conference 3 (1988) pp. 55–65.

4. A.A. Tseng, S. Tong and F.H. Lin, Thermal stresses and expansion of rotating rolls in rolling processing. *Materials in Manufacturing Processes*. New York: ASME, MD 8 (1988) pp. 59–72.
5. P.I. Polukhin, V.M. Rudelson, V.P. Polukhin and V.A. Nikolaev, Izvestia vuzov. *Chernaya Metallurgia* 63 (2) (1974) pp. 82–90.
6. B. Gecim and W.O. Winer, Steady temperature in a rotating cylinder subject to surface heating and convective cooling, *Trans. ASME, J. Tribology* 106 (Jan 1984) pp. 120–127.
7. E.J. Patula, Steady-state temperature distribution in a rotating roll subject to surface heat fluxes and convective cooling. *Trans ASME, J. Heat Transfer* 103 (Feb 1981) pp. 36–43.
8. E.I. Nesis, *Fluid Boiling*. In Russian. Moscow: Nauka (1973).
9. D.M. Parke and J.L.L. Baker, Temperature effects of cooling work rolls, *Iron and Steel Eng*. 49 (2) (1972) pp. 83–88.
10. W.D. Bennon, Evaluation of selective coolant application for the control of work roll thermal expansion, *ASME J. Engineering for Industry* 107 (1985) pp. 146–152.
11. A.A. Tseng, Finite difference solution for heat transfer in a roll rotating at a high speed, *Int. J. Heat Transfer* 7 (1) (Jan–Mar, 1984) pp. 112–124.
12. J.V. Poplavsky and D.A. Seccombe, Jr., Bethlehem's contribution to the mathematical modeling of cold rolling in tandem mills. *Iron and Steel Eng*. 57 (1980) pp. 47–58.
13. S.L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*. In Russian. Novosibirsk (1962).
14. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards. New York: John Wiley & Sons (1972).